A New Formula for The Values of Dirichlet Beta Function at Odd Positive Integers Based on The WZ Method

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Abstract

By using the related results in the WZ theory, a new (as far as I know) formula for the values of Dirichlet beta function $\beta(s) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$ (where $Re(s) > 0$) at odd positive integers was given.

1 Introduction

It is well known that for Riemann Zeta function $\zeta(s) = \sum\limits_{n=1}^{+\infty} \frac{1}{n^s}$, (where $Re(s) > 1$), Dirichlet Lambda function $\lambda(s) = \sum\limits_{n=1}^{+\infty} \frac{1}{(2n-1)^s}$, (where $Re(s) > 1$) and Dirichlet Beta function $\beta(s) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$, (where $Re(s) > 0$), the following formulas are valid

$$\zeta(2n) = \frac{2^{2n-1}(-1)^{n-1}B_{2n}\pi^{2n}}{(2n)!}, \quad n \in N \tag{1}$$

$$\lambda(s) = \frac{2^s - 1}{2^s}\zeta(s), \tag{2}$$

$$\beta(2n+1) = \frac{(-1)^n E_{2n}}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1} \quad , \quad n \in N_0 = N \cup \{0\} \tag{3}$$

where B_n is Bernoulli number, E_n is Euler number, which are given by the following formulas respectively

$$\frac{x}{e^x - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!}x^n, \quad \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{+\infty} \frac{E_n}{n!}x^n.$$

And we know that there are no such simple formulas for $\zeta(2n+1)$ and $\beta(2n)$. In fact, $\zeta(s)$, $\lambda(s)$, $\beta(s)$ are the special cases of Dirichlet L-function $L_k(s) =$

$\sum_{n=1}^{+\infty} \frac{\chi_k(n)}{n^s}$(where $Re(s) > 1$): $\zeta(s) = L_1(s)$, $\lambda(s) = L_2(s)$, $\beta(s) = L_{-4}(s)$, where χ_k is Dirichlet character, see [1]. Some general formulas for $L_{-k}(s)$ and $L_k(s)$ are given in [1], [2] and [3]. It is worth mentioning that the evaluation of special values of Dirichlet L-function is an active research field.

In [4], based on the framework of the WZ theory (see [5], [6], [7]), we obtained a recurrence formula for $\zeta(2l)$

$$\varsigma(2l) = \left(\frac{2^{2l-1}}{1-2^{2l}}\right)\left\{\left[\frac{(-1)^{l+1}}{4l} + \frac{(-1)^l}{2}\right]\frac{\pi^{2l}}{\Gamma(2l)} + \sum_{j=1}^{l-1}\frac{(-1)^{l-j}\pi^{2(l-j)}}{\Gamma(2(l-j)+1)}\varsigma(2j)\right\} \tag{4}$$

where $l \in N$, and $\sum_{k=1}^{0} a(k) = 0$ is a convention, which is equivalent to the following classical formula for Bernoulli polynomial

$$B_{2n}\left(\frac{1}{2}\right) = \left(2^{-2n+1} - 1\right) B_{2n} \tag{5}$$

where $B_n(x)$ is Bernoulli polynomial of order n, given by the following formula:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n(x)}{n!} t^n.$$

In [4], it was pointed out that the ideas of getting the recurrence (4) can be used to evaluate similar infinite series. In this paper, we obtained a new (as far as I know) formula for $\beta(2l-1)$ (where $l \in N$) by using the ideas in [4]. The main steps are given as follows: we obtained the special values of $\beta(2l-1)$ at $l = 1, 2, 3$ by using the method in [4] first, then we formulate a conjecture which give a general formula for $\beta(2l-1)$, finally we proved the conjecture. It is worth mentioning that the method of obtaining $\beta(1)$, $\beta(3)$ and $\beta(5)$ is consistent with the one of proving the conjecture. It is also worth mentioning that when I formulate the conjecture by the special values of $\beta(2l-1)$, I used Mathematica 8.0. For the special values of l, $l = 1, 2, 3$ for example , we can obtain the special values of $\beta(2l-1)$ by Mathematica 8.0.

The main result in this paper is the following theorem.

Theorem. Let $\beta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$, where $Re(s) > 0$, then we have $\beta(1) = \frac{\pi}{4}$, $\beta(3) = \frac{\pi^3}{32}$, $\beta(5) = \frac{5\pi^5}{1536}$, more generally, for all $l \in N$, we have

$$\beta(2l-1) = \frac{(-1)^{l+1}\pi^{2l-1}}{2^{2l}}\left[\frac{1}{\Gamma(2l-1)} + 2\sum_{j=1}^{l-1}\frac{(-1)^j 2^{2j}\lambda(2j)}{\Gamma(2l-2j)\pi^{2j}}\right] \tag{6}$$

$$= \frac{(-1)^{l+1}\pi^{2l-1}}{2^{2l}}\left[\frac{1}{\Gamma(2l-1)} + 2\sum_{j=1}^{l-1}\frac{(-1)^j (2^{2j}-1)\zeta(2j)}{\Gamma(2l-2j)\pi^{2j}}\right] \tag{7}$$

$$= \frac{(-1)^{l+1}\pi^{2l-1}}{2^{2l}} \left[\frac{1}{\Gamma(2l-1)} - \sum_{j=1}^{l-1} \frac{2^{2j}(2^{2j}-1)B_{2j}}{\Gamma(2l-2j)\Gamma(2j+1)} \right]. \tag{8}$$

By formula (3) and formula (8), we obtained the following formula

$$E_{2l} = 1 - \frac{1}{2l+1} \sum_{j=1}^{l} \binom{2l+1}{2j} 2^{2j}(2^{2j}-1)B_{2j} \tag{9}$$

where $l \in N_0$, and $\sum\limits_{k=1}^{0} a(k) = 0$ is a convention.

Both of formula (6) and formula (7) are similar to formula (4). Because formula (7) can be obtained by formula (2) and formula (6), and formula (8) can be obtained by formula (1) and formula (7), we just need to prove formula (6).

2 Preparation Lemmas

To prove the theorem, we need the following lemmas. Lemma 1 can be seen in [4], Lemma 2 can be seen in Page 204 of [8], Lemma 3 can be seen in Page 37 of [9], and the proofs of Lemma 4, Lemma 5 and Lemma 6 will be given below.

Lemma 1. For a continuous-discrete WZ pair $(F(x,k), G(x,k))$, that is, they satisfy the following so-called continuous-discrete WZ equation

$$\frac{\partial F(x,k)}{\partial x} = G(x,k+1) - G(x,k) \tag{10}$$

then for all $m, n \in N_0$, $h, x \in R$, we have

$$\sum_{k=m}^{n} F(x,k) - \sum_{k=m}^{n} F(h,k) = \int_h^x G(t,n+1)dt - \int_h^x G(t,m)dt. \tag{11}$$

Lemma 2. If for all $a, x \in R$, $f(t)$ is integrable on (a,x), then we have

$$\int_a^x \left(\int_a^{t_k} \cdots \left(\int_a^{t_2} f(t_1)dt_1 \right) \cdots dt_{k-1} \right) dt_k = \frac{1}{\Gamma(k)} \int_a^x (x-t)^{k-1} f(t)dt. \tag{12}$$

Lemma 3. For all $n \in N$, we have

$$\sum_{k=1}^{n} \cos((2k-1)x) = \frac{1}{2}\sin(2nx)\csc x.$$

Lemma 4. If $k \in N$, we have

$$\lim_{k\to+\infty} \int_0^{\frac{\pi}{2}} \frac{\sin(2kt)}{\sin t} dt = \frac{\pi}{2}.$$

Proof. At first, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin(2kt)}{\sin t} dt &= \frac{1}{2}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\cot\left(\frac{t}{2}\right) dt \\ &\quad -\frac{1}{2}\int_0^{\pi} \cos\left(\frac{(2k+1)t}{2}\right) dt \\ &= I_1 - I_2,\end{aligned}$$

where

$$I_1 = \frac{1}{2}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\cot\left(\frac{t}{2}\right) dt,$$

$$I_2 = \frac{1}{2}\int_0^{\pi} \cos\left(\frac{(2k+1)t}{2}\right) dt.$$

Now, let us think about I_2 first, because

$$|I_2| = \frac{1}{2}\left|\int_0^{\pi} \cos\left(\frac{(2k+1)t}{2}\right) dt\right| = \frac{1}{2k+1}\left|\sin\left(\frac{(2k+1)\pi}{2}\right)\right| \le \frac{1}{2k+1},$$

we have $\lim\limits_{k\to+\infty} I_2 = 0$. Next, let us think about I_1. Because $\tan\left(\frac{t}{4}\right)$ is continuous on $[0, \pi]$, of course, it is integrable on $[0, \pi]$, by Riemann-Lebesgue Lemma, we have

$$\lim_{k\to+\infty}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\tan\left(\frac{t}{4}\right) dt = 0.$$

By Lemma 4 in [4] and $\csc\left(\frac{t}{2}\right) - \cot\left(\frac{t}{2}\right) = \tan\left(\frac{t}{4}\right)$, we have

$$\lim_{k\to+\infty}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\cot\left(\frac{t}{2}\right) dt = \lim_{k\to+\infty}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\csc\left(\frac{t}{2}\right) dt = \pi.$$

Finally, we have

$$\begin{aligned}\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}} \frac{\sin(2kt)}{\sin t} dt &= \lim_{k\to+\infty}(I_1 - I_2) \\ &= \frac{1}{2}\lim_{k\to+\infty}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\csc\left(\frac{t}{2}\right) dt - \lim_{k\to+\infty} I_2 \\ &= \frac{\pi}{2}.\end{aligned}$$

The proof of Lemma 4 was completed.
Remarks: 1. We can also prove

$$\lim_{k\to+\infty}\int_0^{\pi} \sin\left(\frac{(2k+1)t}{2}\right)\tan\left(\frac{t}{4}\right) dt = 0$$

by The Second Mean Value Theorem for Integrals. Because for all $t \in [0,\pi]$, $\left(\tan\left(\frac{t}{4}\right)\right)' = \frac{1}{4}\left(\sec\left(\frac{t}{4}\right)\right)^2 > 0$, we know that $\tan\left(\frac{t}{4}\right)$ is monotone (increasing) on $[0,\pi]$, we conclude by The Second Mean Value Theorem for Integrals that there exist ξ on $[0,\pi]$, such that

$$\begin{aligned}
&\left|\int_0^\pi \sin\left(\frac{(2k+1)t}{2}\right)\tan\left(\frac{t}{4}\right)dt\right| \\
=& \quad \left|\tan\left(\frac{0+0}{4}\right)\int_0^\xi \sin\left(\frac{(2k+1)t}{t}\right)dt + \tan\left(\frac{\pi-0}{4}\right)\int_\xi^\pi \sin\left(\frac{(2k+1)t}{2}\right)dt\right| \\
\leq& \quad 4\tan\left(\frac{\pi}{4}\right)\frac{1}{2k+1}.
\end{aligned}$$

Finally, we have

$$\lim_{k\to+\infty}\int_0^\pi \sin\left(\frac{(2k+1)t}{2}\right)\tan\left(\frac{t}{4}\right)dt = 0.$$

2. In fact, this Lemma can be proved in a simpler way by using Lemma 3, the idea here is similar to that in Page 66 of [10]. By Lemma 3, for the integrand we have

$$\frac{\sin(2kt)}{\sin(t)} = 2\sum_{k=1}^{n}\cos((2k-1)t).$$

Integrate the sum termwise, and note that

$$\int_0^{\frac{\pi}{2}}\cos((2i-1)t)dt = \frac{(-1)^{i+1}}{2i-1},$$

then we have

$$\int_0^{\frac{\pi}{2}}\frac{\sin(2kt)}{\sin(t)}dt = 2\sum_{i=1}^{k}\frac{(-1)^{i+1}}{2i-1},$$

which tends to $2\arctan(1) = \frac{\pi}{2}$. This completes the proof.
Lemma 5. If $k \in N$, for all $s \geq 1$, we have

$$\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}}\frac{t^s\sin(2kt)}{\sin t}dt = 0.$$

Proof. Let $f(t) = \begin{cases} \frac{t^s}{\sin t} & 0 < t \leq \frac{\pi}{2} \\ 0 & t = 0 \end{cases}$, where $s \geq 1$, then it is easy to prove that $f(t)$ is monotone (increasing) on $\left[0,\frac{\pi}{2}\right]$ (the details of proving will be given in the Remarks below) . By The Second Mean Value Theorem for Integrals, we know that there exist ξ on $\left[0,\frac{\pi}{2}\right]$, such that

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t} \sin(2kt)dt &= \int_0^{\frac{\pi}{2}} f(t)\sin(2kt)dt \\
&= f(0+0)\int_0^{\xi} \sin(2kt)dt + f\left(\frac{\pi}{2}-0\right)\int_{\xi}^{\frac{\pi}{2}} \sin(2kt)dt \\
&= \left(\frac{\pi}{2}\right)^s \int_{\xi}^{\frac{\pi}{2}} \sin(2kt)dt \\
&= \left(\frac{\pi}{2}\right)^s \left[\frac{1}{2k}\left(-\cos(2kt)\right)\Big|_{\xi}^{\frac{\pi}{2}}\right].
\end{aligned}
$$

We conclude that

$$\left|\int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t}\sin(2kt)dt\right| \le \left(\frac{\pi}{2}\right)^s \frac{1}{k},$$

finally, we have

$$\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t}\sin(2kt)dt = 0.$$

The proof of Lemma 5 was completed.

Remarks: 1. In this remark, we give the proof that $f(t)$ is monotone (increasing) on $\left[0, \frac{\pi}{2}\right]$. Because $s \ge 1$, it is easy to prove that $f(t)$ is differentiable on $\left[0, \frac{\pi}{2}\right]$, and $f^{'}(t) = \begin{cases} st^{s-1}\csc t - t^s \cot t \csc t & 0 < t \le \frac{\pi}{2} \\ 0 & t = 0 \end{cases}$. Let $g(t) = s\sin t - t\cos t, \quad 0 \le t \le \frac{\pi}{2}$, then we have $g(0) = 0$, $g^{'}(t) = (s-1)\cos t + t\sin t$. Because $s \ge 1$, for $0 < t \le \frac{\pi}{2}$, we conclude that $(s-1)\cos t \ge 0$ and $t\sin t > 0$, that is, for $0 < t \le \frac{\pi}{2}$, $g^{'}(t) > 0$. We conclude that for $0 < t \le \frac{\pi}{2}$, $g(t) > g(0) = 0$, that is, for $0 < t \le \frac{\pi}{2}$, $f^{'}(t) > f^{'}(0) = 0$. Finally, because $f(t)$ is continuous on $\left[0, \frac{\pi}{2}\right]$, we conclude that $f(t)$ is monotone (increasing) on $\left[0, \frac{\pi}{2}\right]$.

2. In fact, the simpler proof of Lemma 5 is the proof by using Riemann-Lebesgue Lemma directly. Because $f(t) = \begin{cases} \frac{t^s}{\sin t} & 0 < t \le \frac{\pi}{2} \\ 0 & t = 0 \end{cases}$, where $s \ge 1$, is continuous on $\left[0, \frac{\pi}{2}\right]$, of course, it is integrable on $\left[0, \frac{\pi}{2}\right]$, by Riemann-Lebesgue Lemma, we conclude that

$$\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}} f(t)\sin(2kt)dt = 0.$$

Finally, because $\int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t}\sin(2kt)dt = \int_0^{\frac{\pi}{2}} f(t)\sin(2kt)dt$, we conclude that

$$\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t}\sin(2kt)dt = 0.$$

3. By the proof of this lemma above, we conclude that for $\lambda \in R$, the following result is still valid for all $s \geq 1$

$$\lim_{\lambda \to +\infty} \int_0^{\frac{\pi}{2}} \frac{t^s}{\sin t} \sin(\lambda t) dt = 0.$$

4. Similar to Lemma 5 in [4], when $s \geq 2$, we can also prove this lemma by using integration by parts, but when $1 \leq s < 2$, this method can't be used.

Lemma 6. Let $I_l(x,k) = \sum\limits_{j=1}^{k} \frac{\sin((2j-1)x)}{(2j-1)^l}$, $J_l(x,k) = \sum\limits_{j=1}^{k} \frac{\cos((2j-1)x)}{(2j-1)^l}$, where $k, l \in N$, then we have

$$I_{2l+1}(x,k) = \int_0^x \left(\int_0^t -I_{2l-1}(t_1,k) dt_1 \right) dt + \int_0^x J_{2l}(0,k) dt \tag{13}$$

$$\begin{aligned} I_{2l-1}(x,k) &= (-1)^{l+1} \int_0^x \left(\int_0^{t_{2l-1}} \cdots \left(\int_0^{t_2} \frac{\sin(2kt_1)}{2\sin t_1} dt_1 \right) \cdots dt_{2l-2} \right) dt_{2l-1} \\ &\quad + \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1} J_{2j}(0,k) x^{2l-2j-1}}{\Gamma(2l-2j)} & (14) \\ &= \frac{(-1)^{l+1}}{\Gamma(2l-1)} \int_0^x \frac{(x-t)^{2l-2} \sin(2kt)}{2\sin t} dt \\ &\quad + \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1} J_{2j}(0,k) x^{2l-2j-1}}{\Gamma(2l-2j)} & (15) \end{aligned}$$

Proof. Formula (13) will be proved first. Let

$$F_{2l+1}(x,k) = \frac{\sin((2k-1)x)}{(2k-1)^{2l+1}}, \quad G_{2l+1}(x,k) = \sum_{j=1}^{k-1} \frac{\cos((2j-1)x)}{(2j-1)^{2l}}$$

$$F_{2l}(x,k) = \frac{\cos((2k-1)x)}{(2k-1)^{2l}}, \quad G_{2l}(x,k) = \sum_{j=1}^{k-1} \frac{-\sin((2j-1)x)}{(2j-1)^{2l-1}}$$

then it is easy to verify that for $i = 0, 1$, $(F_{2l+i}(x,k), G_{2l+i}(x,k))$ is a WZ pair, by Lemma 1, we have

$$\sum_{j=1}^{k} F_{2l+i}(x,j) - \sum_{j=1}^{k} F_{2l+i}(0,j) = \int_0^x G_{2l+i}(t,k+1) dt - \int_0^x G_{2l+i}(t,1) dt.$$

Because (A) for all j, $F_{2l}(0,j) = \frac{1}{(2j-1)^{2l}}$, $F_{2l+1}(0,j) = 0$, (B) for all t, $G_{2l+i}(t,1) = 0$ by the convention: $\sum\limits_{k=1}^{0} a_k = 0$, (C) it is also easy to verify that

$$G_{2l+1}(t,k+1)=\sum_{j=1}^{k}F_{2l}(x,j),\quad G_{2l}(t,k+1)=-I_{2l-1}(t,k+1),$$

we have

$$\begin{aligned}\sum_{j=1}^{k}F_{2l+1}(x,j) &= \int_0^x G_{2l+1}(t,k+1)dt\\ &= \int_0^x\sum_{j=1}^{k}F_{2l}(t,j)dt\\ &= \int_0^x\left(\int_0^t G_{2l}(t_1,k+1)dt_1\right)dt+\int_0^x J_{2l}(0,k)dt\\ &= \int_0^x\left(\int_0^t -I_{2l-1}(t_1,k+1)dt_1\right)dt+\int_0^x J_{2l}(0,k)dt.\end{aligned}$$

The proof of formula (13) is completed.

Now, we will prove formula (14) by mathematical induction. At first, we will prove that when $l=1,2$, formula (14) is valid. When $l=1$, let

$$F_1(x,k)=\frac{\sin((2k-1)x)}{2k-1},\quad G_1(x,k)=\sum_{j=1}^{k-1}\cos((2j-1)x),$$

it is easy to verify that $(F_1(x,k),G_1(x,k))$ is a WZ pair, by Lemma 1, we have

$$\sum_{j=1}^{k}F_1(x,j)-\sum_{j=1}^{k}F_1(0,j)=\int_0^x G_1(t,k+1)dt-\int_0^x G_1(t,1)dt$$

Because (A) for all j, $F_1(0,j)=0$, (B) for all t, $G_1(t,1)=0$ by the convention: $\sum\limits_{k=1}^{0}a_k=0$, by Lemma 3, we have

$$\begin{aligned}I_1(x,k) &= \sum_{j=1}^{k}F_1(x,j)=\int_0^x G_1(t,k+1)dt\\ &= \int_0^x\sum_{j=1}^{k}\cos((2j-1)t)dt=\frac{1}{2}\int_0^x\frac{\sin(2kt)}{\sin t}dt.\end{aligned}$$

By the convention: $\sum\limits_{k=1}^{0}a_k=0$, we have

$$\sum_{j=1}^{0}\frac{(-1)^jJ_{2j}(0,k)}{\Gamma(2-2j)}x^{1-2j}=0.$$

When $l = 1$, we have

$$
\begin{aligned}
& (-1)^{l+1}\int_0^x\left(\int_0^{t_{2l-1}}\cdots\left(\int_0^{t_2}\frac{\sin(2kt_1)}{2\sin t_1}dt_1\right)\cdots dt_{2l-2}\right)dt_{2l-1} \\
= \quad & (-1)^{1+1}\int_0^x\frac{\sin(2kt_1)}{2\sin t_1}dt_1 = \frac{1}{2}\int_0^x\frac{\sin(2kt)}{\sin t}dt.
\end{aligned}
$$

When $l = 1$, the proof of formula (14) is completed. When $l = 2$, let

$$F_3(x,k) = \frac{\sin((2k-1)x)}{(2k-1)^3}, \quad G_3(x,k) = \sum_{j=1}^{k-1}\frac{\cos((2j-1)x)}{(2j-1)^2}$$

$$F_2(x,k) = \frac{\cos((2k-1)x)}{(2k-1)^2}, \quad G_2(x,k) = \sum_{j=1}^{k-1}\frac{-\sin((2j-1)x)}{2j-1}$$

$$F_1(x,k) = \frac{-\sin((2k-1)x)}{2k-1}, \quad G_1(x,k) = \sum_{j=1}^{k-1}-\cos((2j-1)x),$$

It is easy to verify that for $i = 1, 2, 3$, $(F_i(x,k), G_i(x,k))$ is a WZ pair, by Lemma 1, we have

$$\sum_{j=1}^{k}F_i(x,j) - \sum_{j=1}^{k}F_i(0,j) = \int_0^x G_i(t,k+1)dt - \int_0^x G_i(t,1)dt.$$

Because (A) for all j, $F_3(0,j) = 0$, $F_2(0,j) = \frac{1}{(2j-1)^2}$, $F_1(0,j) = 0$, (B) for all t, $G_i(t,1) = 0$, $i = 1, 2, 3$, by the convention: $\sum\limits_{k=1}^{0} a_k = 0$, (C) it is also easy to verify that

$$G_{i+1}(t,k+1) = \sum_{j=1}^{k}F_i(x,j), \quad i = 1, 2,$$

finally, by Lemma 3, we have

$$
\begin{aligned}
I_3(x,k) \quad &= \quad \sum_{j=1}^{k}F_3(x,j) \\
&= \quad \int_0^x G_3(t,k+1)dt \\
&= \quad \int_0^x\sum_{j=1}^{k}F_2(t,j)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^x \left(\int_0^{t_2} G_2(t_1,k+1)dt_1\right)dt_2 + \int_0^x J_2(0,k)dt \\
&= \int_0^x \left(\int_0^{t_2} \sum_{j=1}^{k} F_1(t_1,j)dt_1\right)dt_2 + \int_0^x J_2(0,k)dt \\
&= \int_0^x \left(\int_0^{t_3} \left(\int_0^{t_2} G_1(t_1,k+1)dt_1\right)dt_2\right)dt_3 + \int_0^x J_2(0,k)dt \\
&= \int_0^x \left(\int_0^{t_3} \left(\int_0^{t_2} \sum_{j=1}^{k} -\cos((2j-1)t_1)dt_1\right)dt_2\right)dt_3 \\
&\quad + \int_0^x J_2(0,k)dt \\
&= -\int_0^x \left(\int_0^{t_3} \left(\int_0^{t_2} \frac{\sin(2kt_1)}{\sin t_1}dt_1\right)dt_2\right)dt_3 + J_2(0,k)x.
\end{aligned}
$$

When $l=2$, the proof of formula (14) is completed. Next, we will prove that with the assumption that formula (14) is valid for l , then formula (14) is also valid for $l+1$. By formula (13) and the assumption above, we have

$$
\begin{aligned}
&I_{2(l+1)-1}(x,k) \\
&\quad = I_{2l+1}(x,k) \\
&\quad = \int_0^x \left(\int_0^{t_{2l+1}} -I_{2l-1}(t_{2l},k)dt_{2l}\right)dt_{2l+1} + \int_0^x J_{2l}(0,k)dt \\
&\quad = (-1)^{l+2}\int_0^x \left(\int_0^{t_{2l+1}} \cdots \left(\int_0^{t_2} \frac{\sin(2kt_1)}{2\sin t_1}dt_1\right)\cdots dt_{2l}\right)dt_{2l+1} \\
&\qquad + \int_0^x \left(\int_0^{t_2} \sum_{j=1}^{l-1} \frac{(-1)^{l+j}J_{2j}(0,k)}{\Gamma(2l-2j)}dt_1\right)dt_2 + \int_0^x J_{2l}(0,k)dt \\
&\quad = (-1)^{l+2}\int_0^x \left(\int_0^{t_{2l+1}} \cdots \left(\int_0^{t_2} \frac{\sin(2kt_1)}{2\sin t_1}dt_1\right)\cdots dt_{2l}\right)dt_{2l+1} \\
&\qquad + \sum_{j=1}^{l-1} \frac{(-1)^{l+j}J_{2j}(0,k)}{(2l-2j+1)(2l-2j)\Gamma(2l-2j)}x^{2l+2j+1} + J_{2l}(0,k)x \\
&\quad = (-1)^{(l+1)+1}\int_0^x \left(\int_0^{t_{2l+1}} \cdots \left(\int_0^{t_2} \frac{\sin(2kt_1)}{2\sin t_1}dt_1\right)\cdots dt_{2l}\right)dt_{2l+1} \\
&\qquad + \sum_{j=1}^{(l+1)-1} \frac{(-1)^{(l+1)+j-1}J_{2j}(0,k)}{\Gamma(2(l+1)-2j)}x^{2(l+1)-2j-1}.
\end{aligned}
$$

We proved that when formula (14) is valid for l , formula (14) is also valid for $l+1$. Finally, by the principle of mathematical induction, we proved that

formula (14) is valid for all $l \in n$. By formula (13) and Lemma 2, it is easy to prove formula (15). The proof of Lemma 6 is completed.

3 Proof of Theorem

As mentioned above, we just need to prove formula (6). By Lemma 6, for all $l \in N$, we have

$$\beta(2l-1) = \lim_{k\to+\infty} I_{2l-1}\left(\frac{\pi}{2}, k\right), \quad \lambda(2l) = \lim_{k\to+\infty} J_{2l}(0,k).$$

By formula (15), we have

$$\begin{aligned} I_{2l-1}(x,k) &= \frac{(-1)^{l+1}}{\Gamma(2l-1)} \int_0^x (x-t)^{2l-2} \frac{\sin(2kt)}{2\sin t} dt \\ &\quad + \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1} J_{2j}(0,k)}{\Gamma(2l-2j)} x^{2l-2j-1} \\ &= I_{2l-1,1}(x,k) + I_{2l-1,2}(x,k), \end{aligned}$$

where

$$I_{2l-1,1}(x,k) = \frac{(-1)^{l+1}}{\Gamma(2l-1)} \int_0^x (x-t)^{2l-2} \frac{\sin(2kt)}{2\sin t} dt,$$

$$I_{2l-1,2}(x,k) = \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1} J_{2j}(0,k)}{\Gamma(2l-2j)} x^{2l-2j-1}.$$

It is easy to see that

$$\begin{aligned} \lim_{k\to+\infty} I_{2l-1,2}\left(\frac{\pi}{2}, k\right) &= \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1}}{\Gamma(2l-2j)} \left(\frac{\pi}{2}\right)^{2l-2j-1} \lim_{k\to+\infty} J_{2l}(0,k) \\ &= \sum_{j=1}^{l-1} \frac{(-1)^{l+j+1}}{\Gamma(2l-2j)} \left(\frac{\pi}{2}\right)^{2l-2j-1} \lambda(2j). \end{aligned}$$

By Lemma 4 and Lemma 5, we have

$$\begin{aligned} &\lim_{k\to+\infty} I_{2l-1,1}\left(\frac{\pi}{2}, k\right) \\ &= \lim_{k\to+\infty} \frac{(-1)^{l+1}}{\Gamma(2l-1)} \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - t\right)^{2l-2} \frac{\sin(2kt)}{2\sin t} dt \\ &= \frac{(-1)^{l+1}}{2\Gamma(2l-1)} \sum_{j=1}^{2l-2} \binom{2l-2}{j} \left(\frac{\pi}{2}\right)^{2l-2-j} \lim_{k\to+\infty} \int_0^{\frac{\pi}{2}} \frac{t^j \sin(2kt)}{\sin t} dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^{l+1}}{2\Gamma(2l-1)}\left(\frac{\pi}{2}\right)^{2l-2}\lim_{k\to+\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2kt)}{\sin t}dt\\
= \quad &\frac{(-1)^{l+1}\pi^{2l-1}}{2^{2l}\Gamma(2l-1)}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\beta(2l-1)\\
&= \lim_{k\to+\infty} I_{2l-1}\left(\frac{\pi}{2},k\right)\\
&= \lim_{k\to+\infty} I_{2l-1,1}\left(\frac{\pi}{2},k\right)+\lim_{k\to+\infty} I_{2l-1,2}\left(\frac{\pi}{2},k\right)\\
&= \frac{(-1)^{l+1}\pi^{2l-1}}{2^{2l}}\left[\frac{1}{\Gamma(2l-1)}+2\sum_{j=1}^{l-1}\frac{(-1)^j 2^{2j}\lambda(2j)}{\Gamma(2l-2j)\pi^{2j}}\right].
\end{aligned}
$$

The proof of Theorem is completed.

References

[1] G. Shimura, Elementary Dirichlet Series and Modular Forms, New York: Springer-Verlag, 2007

[2] I. J. Zuker, M. M. Robertson, Some Properties of Dirichlet L-Series, J. Phys. A 9(1976), 1207-1214

[3] I. J. Zuker, R. C. McPhedran, Dirichlet L-Series with real and complex characters and their application to solving double sums, Proc. Roy Soc. A 464(2008), 1405-1422

[4] Y. J. Chen, Evaluations of $\varsigma(2)$, $\varsigma(4)$, $\ldots$, $\varsigma(2k)$ based on the WZ method, available at `http://arxiv.org/pdf/1204.4162v1.pdf`

[5] H. S. Wilf, D. Zeilberger, Rational functions certify combinatorial identities, J. of Amer. Math. Soc. 3(1990) ,147-158

[6] M. Petkovšek, H. S. Wilf, D.Zeilberger, A=B, MA: A.K. Peters Wellesly , 1996

[7] D. Zeilberger, Closed form (Pun intended!), Contemp. Math. 143 (1993) (579-607)

[8] Z. A. Melzak, Companion to Concrete Mathematics: Mathematical Techniques and Various Applications, New York: John Wiley&Sons,Inc. , 1973

[9] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products, 7th ed. New York: Academic Press 2007

[10] William Elwood Byerly, An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, With Applications to Problems in Mathematical Physics, New York: Ginn&Company, 1893

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